

Constant Factor Lasserre Integrality Gaps for Graph Partitioning Problems

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Abstract

Partitioning the vertices of a graph into two roughly equal parts while minimizing the number of edges crossing the cut is a fundamental problem (called Balanced Separator) that arises in many settings. For this problem, and variants such as the Uniform Sparsest Cut problem where the goal is to minimize the fraction of pairs on opposite sides of the cut that are connected by an edge, there are large gaps between the known approximation algorithms and non-approximability results. While no constant factor approximation algorithms are known, even APX-hardness is not known either.

In this work we prove that for balanced separator and uniform sparsest cut, semidefinite programs from the Lasserre hierarchy (which are the most powerful relaxations studied in the literature) have an integrality gap bounded away from 1, even for $\Omega(n)$ levels of the hierarchy. This complements recent algorithmic results in [GS11] which used the Lasserre hierarchy to give an approximation scheme for these problems (with runtime depending on the spectrum of the graph). Along the way, we make an observation that simplifies the task of lifting “polynomial constraints” (such as the global balance constraint in balanced separator) to higher levels of the Lasserre hierarchy.

We also obtain a similar result for Max Cut and prove that even linear number of levels of the Lasserre hierarchy have an integrality gap exceeding $18/17 - o(1)$, though in this case there are known NP-hardness results with this gap.

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1 Introduction

Partitioning a graph into two (balanced) parts with few edges going across them is a fundamental optimization problem. Graph partitions or separators are widely used in many applications (such as clustering, divide and conquer algorithms, VLSI layout, etc). Two prototypical objectives of graph partitioning are **Balanced Separator** and **Uniform Sparsest Cut**, defined as follows.

Definition 1.1. *Given an undirected graph $G = (V, E)$ and $0 < \tau < 0.5$, the goal of the τ vs $1 - \tau$ **Balanced Separator** problem is to find a set $A \subseteq V$ such that $\tau|V| \leq |A| \leq (1 - \tau)|V|$, while $\text{edges}(A, V \setminus A)$ is minimized. Here $\text{edges}(A, B)$ is the number of edges in E that cross the cut (A, B) .*

*The goal of the **Uniform Sparsest Cut** problem is to find a set $\emptyset \subsetneq A \subsetneq V$ such that the sparsity*

$$\frac{\text{edges}(A, V \setminus A)}{|A||V \setminus A|}$$

is minimized.

The two problems are intensively studied in both approximation and hardness of approximation side, but there are still huge gaps between the known upper bounds and lower bounds. The best algorithms, based on semidefinite relaxations (SDPs) with triangle inequalities, gives a $O(\sqrt{\log n})$ approximation to both problems [ARV04]. On the inapproximability side, a Polynomial Time Approximation Scheme (PTAS) is ruled out for both problems assuming NP does not have randomized subexponential-time algorithms [AMS07]. We note that in this paper, we focus on the **Uniform Sparsest Cut** problem – while the general **Sparsest Cut** problem is shown not to have constant-factor approximation algorithm assuming the Unique Games Conjecture (UGC) [CKK⁺06, KV05, Kho02].

It is known that the SDP used in [ARV04] cannot give a constant factor approximation for **Uniform Sparsest Cut** [DKSV06]. Indeed, integrality gaps are shown even for some stronger SDPs: super-constant factor integrality gaps for both **Balanced Separator** and **Uniform Sparsest Cut** are known for the so-called *Sherali-Adams₊ hierarchy* for a super-constant number of rounds [RS09].

However, the potential of an even more powerful form of SDPs, called the Lasserre hierarchy, is not well understood. Indeed, it is consistent with current knowledge that even 4 rounds of the Lasserre hierarchy could improve the factor 0.878 Goemans-Williamson algorithm for **Maximum Cut**, and therefore refute the Unique Games conjecture. For the graph partitioning problems of interest in this paper (**Balanced Separator** and **Uniform Sparsest Cut**), integrality gaps were not known even for a small constant number of rounds. It was not ruled out, for example, that $1/\epsilon^{O(1)}$ rounds of the hierarchy could give a $(1 + \epsilon)$ -approximation algorithm, thereby giving a PTAS. On the algorithmic side, [GS11] recently showed that for these problems, SDPs using $O(r/\epsilon^2)$ rounds of the Lasserre hierarchy have an integrality gap at most $(1 + \epsilon)/\min\{1, \lambda_r\}$. Here λ_r is the r -th smallest eigenvalue of the normalized Laplacian of the graph. This result implies an approximation scheme for these problems with runtime parameterized by the graph spectrum.

Given this situation, it is natural to study the limitations of the Lasserre hierarchy for these two fundamental graph partitioning problems. Several of the known results on strong integrality gap results for many rounds of the Lasserre hierarchy, starting with Schoenebeck’s remarkable construction [Sch08], apply in situations where a corresponding NP-hardness result is already known. Thus they are not “prescriptive” of hardness. In fact, we are aware of only the following examples

where a polynomial-round Lasserre integrality gap stronger than the corresponding NP-hardness result is known: Max k -CSP, k -coloring [Tul09] and Densest k -Subgraph [BCG⁺12]. The main results of this paper, described next, extend this body of results, by showing that Lasserre SDPs cannot give a PTAS for Balanced Separator and Uniform Sparsest Cut.

1.1 Our results

In this paper, we study integrality gaps for the Lasserre SDP relaxations for Balanced Separator and Uniform Sparsest Cut. As mentioned before, APX-hardness is not known for these two problems, even assuming the Unique Games conjecture. (Superconstant hardness results are known based on a strong intractability assumption concerning the Small Set Expansion problem [RST10].) In contrast, we show that linear-round Lasserre SDP has an integrality gap bounded away from 1, and thus fails to give a factor α -approximation for some absolute constant $\alpha > 1$. Specifically, we prove the following two theorems.

Theorem 1.2 (informal). *For $0.45 < \tau < 0.5$, there are linear-round Lasserre gap instances for the τ vs $(1 - \tau)$ Balanced Separator problem, such that the integral optimal solution is at least $(1 + \epsilon(\tau))$ times the SDP solution, where $\epsilon(\tau) > 0$ is a constant dependent on τ .*

Theorem 1.3 (informal). *There are linear-round Lasserre gap instances for the Uniform Sparsest Cut problem, such that the integral optimal solution is at least $(1 + \epsilon)$ times the SDP solution, for some constant $\epsilon > 0$.*

We also study the Lasserre relaxation for Maximum Cut. As mentioned above, we are not able to rule out the possibility that Lasserre SDPs can beat the 0.878-approximation factor (achieved by the Goemans-Williamson algorithm). In [GS11], the authors got a factor $(1 + \epsilon) / \min\{1, (2 - \lambda_{n-r})\}$ -approximation algorithm for Minimum Uncut using $O_\epsilon(r)$ rounds of Lasserre hierarchy. Here we show a $17/18$ gap for $\Omega(n)$ rounds Lasserre SDP, as follows.

Theorem 1.4 (informal). *There are linear-round Lasserre gap instances for the Maximum Cut problem, such that the integral optimal solution is at most $17/18 + o(1)$ times the SDP solution.*

The above result for Maximum Cut is not surprising given the factor $16/17 + \epsilon$ NP-hardness [Hås01, TSSW00]. Thus, for Maximum Cut, we are not able to prove a result stronger than the known NP-hardness results. But the result, to the best of our knowledge, was not known before. (With use of better gadgets we can presumably match the $16/17$ factor also in our gap, but we choose to present a simple gadget with the weaker bound.)

1.2 Our techniques

All of our gap results are based on Schoenebeck's ingenious Lasserre integrality gap for 3-XOR [Sch08]. For Balanced Separator and Uniform Sparsest Cut, we use the ideas in [AMS07] to build gadget reductions and combine them with Schoenebeck's gap instance. [AMS07] designed gadget reductions from Khot's quasi-random PCP [Kho06] in order to show APX-Hardness of the two problems. If we view the Lasserre hierarchy as a computational model (as suggested in [Tul09]), we can view Schoenebeck's construction as playing the role of a quasi-random PCP in the Lasserre model. Our gadget reductions, therefore, bear some resemblance to the ones in [AMS07], though

the analysis is different due to different random structures of the PCPs. We feel our reductions are slightly simpler than the ones in [AMS07], although we need some additional tricks to make the reductions have only linear blowup. This latter feature is needed in order to get Lasserre SDP gaps for a linear number of rounds.

Also, unlike 3-XOR, for balanced separator there is a global linear constraint (stipulating the balance of the cut), and our Lasserre solution must also satisfy a lifted form of this constraint [Las02]. We make a general observation that such constraints can be easily lifted to the Lasserre hierarchy when the vectors in our construction satisfy a related linear constraint. This observation applies to constraints given by any polynomials, and to our knowledge, was not made before. It simplifies the task of constructing legal Lasserre vectors in such cases.

For Maximum Cut, we use a simple gadget reduction from the Monotone 4-XOR problem to get the $\frac{17}{18}$ -factor gap instances. A more sophisticated reduction (the one reducing 3-XOR to Max Cut from [TSSW00]) will probably yield $\frac{16}{17}$ -factor gap instances and match the currently known NP-Hardness, but we choose to present a simple gadget in this version of the paper.

2 Lasserre SDPs

In this section, we begin with a general description of semidefinite programming relaxations from the Lasserre hierarchy, followed by a useful observation about constructing feasible solutions for such a SDP. We then discuss the specific SDP relaxations for our problems of interest. Finally, we recall Schoenebeck's Lasserre integrality gaps [Sch08] in a form convenient for our later use.

2.1 Lasserre Hierarchy Relaxation

Consider a binary programming problem with a single constraint expressed as a polynomial:

$$\begin{aligned} & \text{Minimize/Maximize} && \sum_{T \in \binom{[n]}{\leq d}} P_j(T) \prod_{j \in T} x_j \\ & \text{subject to} && \sum_{T \in \binom{[n]}{\leq d}} Q(T) \prod_{j \in T} x_j \geq 0, \\ & && x_i \in \{0, 1\} \quad \text{for all } i \in [n]. \end{aligned} \tag{1}$$

It is easy to see that this captures all problems we consider in this paper: Balanced Separator (Section 2.2.3), Uniform Sparsest Cut (Section 2.2.2) and Maximum Cut (Section 2.2.3).

Lemma 2.1. *For any positive integer $r \geq d$, r rounds of Lasserre Hierarchy relaxation [Las02] of (1) is given by the following semidefinite programming formulation:*

$$\begin{aligned} & \text{Minimize/Maximize} && \sum_T P_j(T) \|\bar{\mathbf{U}}_T\|^2 \\ & \text{subject to} && \|\bar{\mathbf{U}}_\emptyset\|^2 = 1, \\ & && \langle \bar{\mathbf{U}}_A, \bar{\mathbf{U}}_B \rangle = \|\bar{\mathbf{U}}_{A \cup B}\|^2 \quad \text{for all } A, B \text{ with } |A \cup B| \leq 2r, \\ & && \left\langle \sum_{S \in \binom{[n]}{\leq d}} Q(S) \bar{\mathbf{U}}_{S \cup A}, \sum_{T \in \binom{[n]}{\leq d}} Q(T) \bar{\mathbf{U}}_{T \cup B} \right\rangle = \langle Y_A, Y_B \rangle, \\ & && \langle Y_A, Y_B \rangle = \|Y_{A \cup B}\|^2 \quad \text{for all } A, B \text{ with } |A \cup B| \leq 2(r-d). \\ & && \bar{\mathbf{U}}_A, Y_B \in \mathbb{R}^{\Upsilon}. \end{aligned} \tag{2}$$

Note that a straightforward verification of last two constraints requires the construction of vectors Y_A in addition to $\bar{\mathbf{U}}_A$. Below we give an easier way to verify these last two constraints

without having to construct Y_A 's. This greatly simplifies our task of constructing Lasserre vectors for the lifting of global balance constraints.

Theorem 2.2. *Given vectors \bar{U}_T for all $T \in \binom{[n]}{\leq 2r}$ satisfying the first two constraints of (2), if there exists a non-negative real $\delta > 0$ such that*

$$\sum_{S \in \binom{[n]}{\leq d}} Q(S) \bar{U}_S = \delta \cdot \bar{U}_\emptyset \quad (3)$$

then these vectors form (part of) a feasible solution.

Proof. Consider the following vectors. For each A with $|A| \leq r$, let $Y_A = \sum_S Q(S) \bar{U}_{S \cup A}$. By construction, these vectors satisfy the third constraint. Now we will prove that, for any A, B , $\langle Y_A, Y_B \rangle = \delta^2 \|\bar{U}_{A \cup B}\|^2$ which implies that these vectors satisfy the final constraint as well, completing our proof.

$$\begin{aligned} \langle Y_A, Y_B \rangle &= \left\langle \sum_{S \in \binom{[n]}{\leq d}} Q(S) \bar{U}_{S \cup A}, \sum_{T \in \binom{[n]}{\leq d}} Q(T) \bar{U}_{T \cup B} \right\rangle = \sum_S Q(S) \left\langle \bar{U}_{S \cup A}, \sum_T Q(T) \bar{U}_{T \cup B} \right\rangle \\ &= \sum_{S, T} Q(S) Q(T) \langle \bar{U}_{S \cup A}, \bar{U}_{T \cup B} \rangle = \sum_{S, T} Q(S) Q(T) \langle \bar{U}_{S \cup A \cup B}, \bar{U}_T \rangle \\ &= \sum_S Q(S) \langle \bar{U}_{S \cup A \cup B}, \sum_T Q(T) \bar{U}_T \rangle = \sum_S Q(S) \langle \bar{U}_{S \cup A \cup B}, \delta \bar{U}_\emptyset \rangle \\ &= \delta \sum_S Q(S) \langle \bar{U}_S, \bar{U}_{A \cup B} \rangle = \delta \left\langle \sum_S Q(S) \bar{U}_S, \bar{U}_{A \cup B} \right\rangle = \delta \langle \delta \bar{U}_\emptyset, \bar{U}_{A \cup B} \rangle \\ &= \delta^2 \langle \bar{U}_\emptyset, \bar{U}_{A \cup B} \rangle = \delta^2 \|\bar{U}_{A \cup B}\|^2. \end{aligned} \quad \square$$

2.2 Lasserre SDP for graph partitioning problems

In light of Theorem 2.2, to show good solutions for the Lasserre SDP for our problems of interest, we only need to show good solutions for the following SDPs.

2.2.1 Balanced Separator

The following is the standard integer programming formulation of Balanced Separator:

$$\begin{aligned} &\text{minimize} \quad \sum_{(u,v) \in E} (x_u - x_v)^2 \\ &\text{s.t.} \quad \tau|V| \leq \sum_{u \in V} x_u \leq (1 - \tau)|V| \\ &\quad \quad x_u \in \{0, 1\} \quad \forall u \in V. \end{aligned}$$

The r round Lasserre SDP relaxation has a vector \bar{U}_S for each subset $S \subseteq V$ with $|S| \leq r$. In an integral solution, the intended value of $\bar{U}_{\{u\}}$ is $x_u \bar{U}_\emptyset$ for some fixed unit vector \bar{U}_\emptyset , and that of \bar{U}_S is $\left(\prod_{u \in S} x_u\right) \bar{U}_\emptyset$. The vectors \bar{U}_S for $|S| \leq r$ are subject to the following constraints.

Definition 2.3 (SDP Relaxation Ψ_1).

$$\begin{aligned}
& \text{minimize} && \sum_{(u,v) \in E} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 \\
& \text{s.t.} && \langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle \geq 0 \text{ for all } S_1, S_2 \\
& && \langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle = \langle \bar{U}_{S_3}, \bar{U}_{S_4} \rangle \text{ for all } S_1 \cup S_2 = S_3 \cup S_4 \\
& && \|\bar{U}_\emptyset\|^2 = 1 \\
& && \sum_v \bar{U}_{\{v\}} = \tau' |V| \bar{U}_\emptyset \text{ for some } \tau \leq \tau' \leq 1 - \tau
\end{aligned}$$

2.2.2 Uniform Sparsest Cut

The Uniform Sparsest Cut problem asks to minimize the value of the following quadratic integer program over all $\tau \in (0, 1)$.

$$\begin{aligned}
& \text{minimize} && \frac{1}{|V|^2 \tau (1 - \tau)} \sum_{(u,v) \in E} (x_u - x_v)^2 \\
& \text{s.t.} && \sum_u x_u = \tau |V| \\
& && x_u \in \{0, 1\} \quad \forall u \in V.
\end{aligned}$$

The Lasserre relaxation is to minimize the value of the following SDP over all $\tau \in (0, 1)$,

Definition 2.4 (Relaxation Ψ_2).

$$\begin{aligned}
& \text{minimize} && \sum_{(u,v) \in E} \frac{1}{|V|^2 \tau (1 - \tau)} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 \\
& \text{s.t.} && \langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle \geq 0 \text{ for all } S_1, S_2 \\
& && \langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle = \langle \bar{U}_{S_3}, \bar{U}_{S_4} \rangle \text{ for all } S_1 \cup S_2 = S_3 \cup S_4 \\
& && \|\bar{U}_\emptyset\|^2 = 1 \\
& && \sum_v \bar{U}_{\{v\}} = \tau |V| \bar{U}_\emptyset
\end{aligned}$$

2.2.3 Maximum Cut

The following is the standard quadratic integer programming formulation of Maximum Cut:

$$\begin{aligned}
& \text{maximize} && \sum_{(u,v) \in E} (x_u - x_v)^2 \\
& \text{s.t.} && x_u \in \{0, 1\} \quad \forall u \in V.
\end{aligned}$$

The r round Lasserre SDP relaxation has a vector \bar{U}_S for each subset $S \subseteq V$ with $|S| \leq r$ with the following constraints.

Definition 2.5 (SDP Relaxation Ψ_3).

$$\begin{aligned}
& \text{maximize} && \sum_{(u,v) \in E} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 \\
& \text{s.t.} && \langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle \geq 0 \text{ for all } S_1, S_2 \\
& && \langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle = \langle \bar{U}_{S_3}, \bar{U}_{S_4} \rangle \text{ for all } S_1 \cup S_2 = S_3 \cup S_4 \\
& && \|\bar{U}_\emptyset\|^2 = 1
\end{aligned}$$

2.3 Lasserre Gaps for 3-XOR and Monotone 4-XOR from [Sch08]

We start by defining the 3-XOR problem.

Definition 2.6. An instance Ψ of 3-XOR is a set of constraints C_1, C_2, \dots, C_m where each constraint C_i is over 3 distinct variables x_{i_1}, x_{i_2} , and x_{i_3} , and is of the form $x_{i_1} \oplus x_{i_2} \oplus x_{i_3} = b_i$ for some $b_i \in \{0, 1\}$.

A random instance of 3-XOR is sampled by choosing each constraint C_i uniform independently from the set of possible constraints.

Definition 2.7. An instance Ψ of Monotone 4-XOR is a set of constraints C_1, C_2, \dots, C_m where each constraint C_i is over 4 distinct variables $x_{i_1}, x_{i_2}, x_{i_3}$ and x_{i_4} , and is of the form $x_{i_1} \oplus x_{i_2} \oplus x_{i_3} \oplus x_{i_4} = 1$.

A random instance of Monotone 4-XOR is sampled by choosing each constraint C_i uniformly and independently from the set of possible constraints.

We will make use of the following fundamental result of Schoenebeck.

Theorem 2.8 ([Sch08]). For every large enough constant $\beta > 1$, there exists $\eta > 0$, such that with probability $1 - o(1)$, a random 3-XOR instance Ψ over $m = \beta n$ constraints and n variables cannot be refuted by the SDP relaxation obtained by ηn rounds of the Lasserre hierarchy, i.e. there are vectors $\mathbf{W}_{(S,\alpha)}$ for all $|S| \leq \eta n$ and all $\alpha : S \rightarrow \{0, 1\}$, such that

- (i) the value of the solution is perfect: $\sum_{i=1}^m \sum_{\alpha: \alpha(x_{i_1}) \oplus \alpha(x_{i_2}) \oplus \alpha(x_{i_3}) = b_i} \|\mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha)}\|^2 = m$;
- (ii) $\langle \mathbf{W}_{(S_1, \alpha_1)}, \mathbf{W}_{(S_2, \alpha_2)} \rangle \geq 0$ for all $S_1, S_2, \alpha_1, \alpha_2$;
- (iii) $\langle \mathbf{W}_{(S_1, \alpha_1)}, \mathbf{W}_{(S_2, \alpha_2)} \rangle = 0$ if $\alpha_1(S_1 \cap S_2) \neq \alpha_2(S_1 \cap S_2)$;
- (iv) $\langle \mathbf{W}_{(S_1, \alpha_1)}, \mathbf{W}_{(S_2, \alpha_2)} \rangle = \langle \mathbf{W}_{(S_3, \alpha_3)}, \mathbf{W}_{(S_4, \alpha_4)} \rangle$ for all $S_1 \cup S_2 = S_3 \cup S_4$ and $\alpha_1 \circ \alpha_2 = \alpha_3 \circ \alpha_4$;
- (v) $\sum_{\alpha: S \rightarrow \{0, 1\}} \|\mathbf{W}_{(S, \alpha)}\|^2 = 1$ for all S .

Note that indeed we have for every S , $\sum_{\alpha: S \rightarrow \{0, 1\}} \mathbf{W}_{(S, \alpha)} = \mathbf{W}_{(\emptyset, \emptyset)}$. This is because $\|\mathbf{W}_{(\emptyset, \emptyset)}\|^2 = 1$ and

$$\left\langle \left(\sum_{\alpha: S \rightarrow \{0, 1\}} \mathbf{W}_{(S, \alpha)} \right), \mathbf{W}_{(\emptyset, \emptyset)} \right\rangle = \sum_{\alpha: S \rightarrow \{0, 1\}} \langle \mathbf{W}_{(S, \alpha)}, \mathbf{W}_{(\emptyset, \emptyset)} \rangle = \sum_{\alpha: S \rightarrow \{0, 1\}} \|\mathbf{W}_{(S, \alpha)}\|^2 = 1.$$

We also state the following version, which also follows from [Sch08], and will be useful for Maximum Cut gap.

Theorem 2.9. *For every large enough constant $\beta > 1$, there exists $\eta > 0$, such that with probability $1 - o(1)$, a random Monotone 4-XOR instance Ψ over $m = \beta n$ constraints and n variables cannot refuted by the SDP relaxation obtained by ηn rounds of the Lasserre hierarchy, i.e., there are vectors $\mathbf{W}_{(S,\alpha)}$ for all $|S| \leq \eta n$ and all $\alpha : S \rightarrow \{0, 1\}$, such that*

- *The solution has perfect value: $\sum_{i=1}^m \sum_{\alpha: \alpha(x_{i_1}) \oplus \alpha(x_{i_2}) \oplus \alpha(x_{i_3}) \oplus \alpha(x_{i_4}) = 1} \left\| \mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}\}, \alpha)} \right\|^2 = m$;*
- *Conditions (ii) to (v) of Theorem 2.8 are met.*

Observation 2.10. *In both of the constructions Theorem 2.8 and Theorem 2.9, the vectors \mathbf{W} satisfy the following property. For any constraint C_i over set of variables S_i , the vectors corresponding to all satisfying partial assignments of S_i sums up to \mathbf{W}_\emptyset :*

$$\sum_{\alpha: S_i \rightarrow \{0,1\} \wedge C_i(\alpha)=1} \mathbf{W}_{(S_i, \alpha)} = \mathbf{W}_\emptyset.$$

3 Gaps for Balanced Separator

In this section, we prove Theorem 1.2. We state the theorem in detail as follows.

Theorem 3.1. *For large enough constant β, M , for all $0.45 < \tau < 0.5$, there is an instance \mathcal{H}_Ψ for τ vs. $(1 - \tau)$ Balanced Separator problem, such that the optimal solution is at least $4(3\tau - \tau^3)/5 + O(1/\beta + 1/M)$ times the best solution of the $\Omega(n)$ -round Lasserre SDP relaxation. Moreover, the solution for Lasserre SDP relaxation is a fractional $(0.5 - O(1/M))$ vs. $(0.5 + O(1/M))$ balanced separator.*

3.1 Reduction

Given a 3-XOR instance Ψ with $m = \beta n$ constraints and n variables, we build a graph $\mathcal{H}_\Psi = (\mathcal{V}_\Psi, \mathcal{E}_\Psi)$ for Balanced Separator as follows.

\mathcal{H}_Ψ consists of an almost bipartite graph $H_\Psi = (L_\Psi, R_\Psi, E_\Psi)$ (obtained by replacing each right vertex of a bipartite graph by a clique), a clique Z_r , and edges between L_Ψ and Z_r .

The left side L_Ψ of H_Ψ contains $4m = 4\beta n$ vertices, each corresponds to a pair of a constraint and a satisfying partial assignment for the constraint, i.e.

$$L_\Psi = \{(C_i, \alpha) | \alpha : \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0, 1\}, C_i(\alpha) = 1\}.$$

The right side R_Ψ of H_Ψ contains $2n$ cliques, each is of size $M\beta$, and corresponds to one of the $2n$ literals, i.e.

$$R_\Psi = \cup_{j, \alpha: \{x_j\} \rightarrow \{0,1\}} C_{(x_j, \alpha)},$$

where

$$C_{(x_j, \alpha)} = \{(x_j, \alpha, t) | 1 \leq t \leq M\beta\}.$$

Call $(x_j, \alpha, 1)$ the *representative vertex* of $C_{(x_j, \alpha)}$. Besides the clique edges, we connect a left vertex (C_i, α) and a right representative vertex $(x_j, \alpha', 1)$ if x_j is accessed by C_i and α' is consistent to α , i.e.

$$E_\Psi = \{\text{clique edges}\} \cup \{(C_i, \alpha), (x_j, \alpha', 1) | x_j \in \{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha(x_j) = \alpha'(x_j)\}.$$

Now we have finished the definition of H_Ψ . To get \mathcal{H}_Ψ , we add a clique Z_r of size $\sqrt{M\beta n}$ for some very large constant M . We connect each vertex in L_Ψ to two different vertices in Z_r , so that each vertex in Z_r has the same number of neighbors in L_Ψ (this number should be $4\beta n \cdot 2/\sqrt{M\beta n} = 8\sqrt{\beta n/M}$). In other words, if we view each vertex in L_Ψ as an undirected edge between its two neighbors in Z_r , the graph should be a regular graph.

The whole construction is shown in Figure 1. Our construction is very similar to the one in [AMS07], but there are some technical differences. Instead of having cliques in R_Ψ , [AMS07] has clusters of vertices with no edges connecting them. Also, in our construction, the vertices in L_Ψ are connected to the representative vertices in R_Ψ only, while in [AMS07], all the vertices in the right clusters could be connected to the left side. The most important difference is that in our way, the cliques are of constant size, while the clusters in [AMS07] has superconstantly many vertices. This means that our reduction blows up the instance size only by a constant factor, therefore we are able to get linear round Lasserre gap.

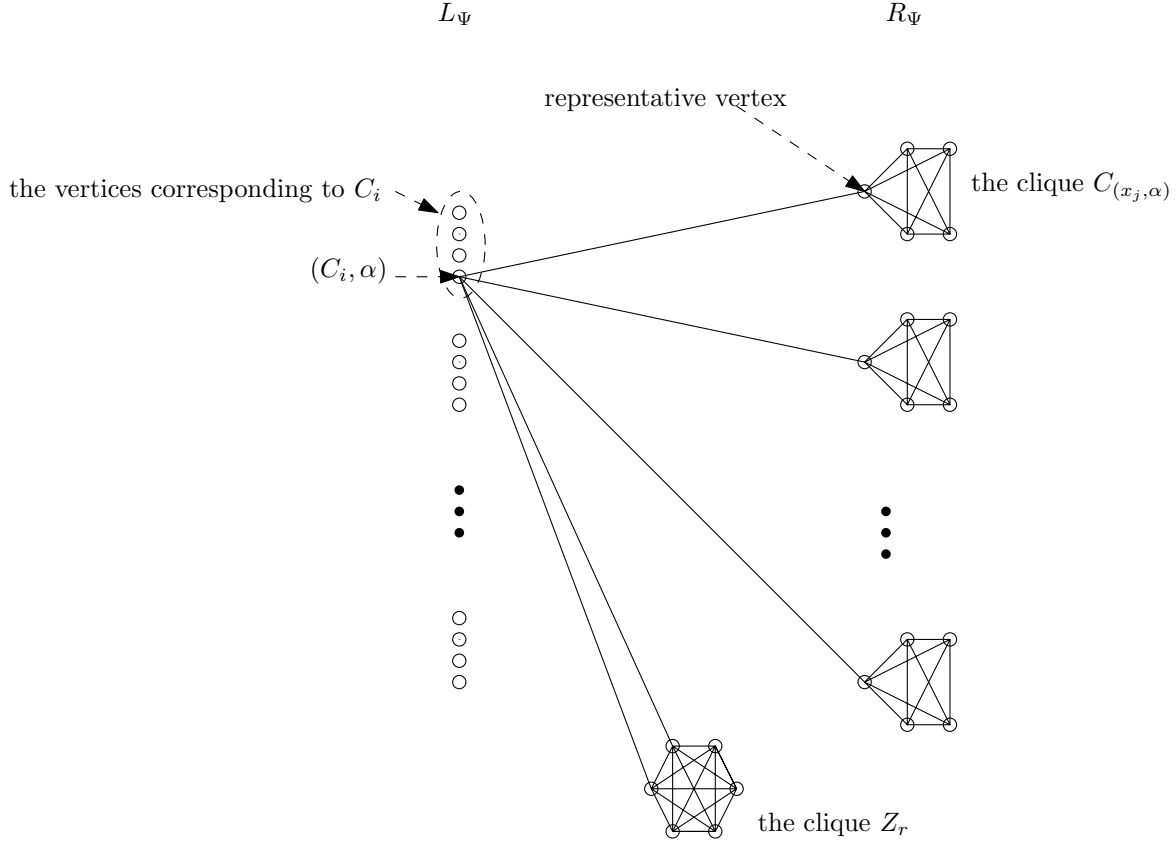


Figure 1: The reduction for Balanced Separator. Note that the incident edges are drawn for only one of the vertices in L_Ψ , while others can be drawn similarly.

Observe that there are $|L_\Psi| + |R_\Psi| + |Z_r| = 4m + 2Mm + \sqrt{Mm} = (2M + 4 + \sqrt{M/m})m = (2M + O(1))m$ vertices in \mathcal{H}_Ψ .

3.2 Completeness : good SDP solution

Lemma 3.2. *If the 3-XOR instance Ψ admits perfect solution for r -round Lasserre SDP relaxation, and satisfies the first condition in Lemma 3.3, then the $r/3$ -round SDP relaxation Ψ_1 (in Definition 2.3) for the Balanced Separator instance \mathcal{H}_Ψ has a solution of value $5m$.*

Proof. For each set $S \subseteq L_\Psi \cup R_\Psi \cup Z_r$ with $|S| \leq r/3$, we define the vector \bar{U}_S as follows. If $S \cap Z_r \neq \emptyset$, let $\bar{U}_S = \mathbf{0}$. If $S \cap Z_r = \emptyset$, suppose that $S \cap L_\Psi$ contains

$$(C_{i_1}, \alpha_1), (C_{i_2}, \alpha_2), \dots, (C_{i_{r_1}}, \alpha_{r_1}),$$

$S \cap R_\Psi$ contains

$$(x_{j_1}, \alpha'_1, t_1), (x_{j_2}, \alpha'_2, t_2), \dots, (x_{j_{r_2}}, \alpha'_{r_2}, t_{r_2}),$$

we have $r_1 + r_2 = |S|$. Let S' be the set of variables accessed by $C_{i_1}, \dots, C_{i_{r_1}}$ together with $x_{j_1}, \dots, x_{j_{r_2}}$. Note that $|S'| \leq 3r_1 + r_2 \leq 3|S| \leq r$. If there is no contradiction among the partial assignments α_i 's and α'_i 's (i.e. there are not two of them assigning the same variable to different values), we can define

$$\alpha = \alpha_1 \circ \alpha_2 \circ \dots \circ \alpha_{r_1} \circ \alpha'_1 \circ \alpha'_2 \circ \dots \circ \alpha'_{r_2}.$$

and let $\bar{U}_S = \mathbf{W}_{(S', \alpha)}$, or we let $\bar{U}_S = \mathbf{0}$.

We first check that the first 3 constraints in relaxation Ψ_1 are satisfied.

- For two sets S_1, S_2 , either at least one of the vectors $\bar{U}_{S_1}, \bar{U}_{S_2}$ is $\mathbf{0}$ (therefore their inner-product is 0), or $\bar{U}_{S_1} = \mathbf{W}_{S'_1, \alpha_1}, \bar{U}_{S_2} = \mathbf{W}_{S'_2, \alpha_2}$ for some $S'_1, S'_2, \alpha_1, \alpha_2$ and $\langle \bar{U}_{S_1}, \bar{U}_{S_2} \rangle = \langle \mathbf{W}_{S'_1, \alpha_1}, \mathbf{W}_{S'_2, \alpha_2} \rangle \geq 0$.
- For any S_1, S_2, S_3, S_4 such that $S_1 \cup S_2 = S_3 \cup S_4$, either the set of partial assignments in $S_1 \cup S_2 = S_3 \cup S_4$ are consistent to each other, in which case we have $\bar{U}_{S_1 \cup S_2} = \bar{U}_{S_3 \cup S_4} = \mathbf{W}_{S, \alpha}$ where S is the union of all the variables included in $S_1 \cup S_2$ and α is the concatenation of the partial assignments in $S_1 \cup S_2$; or we have $\bar{U}_{S_1 \cup S_2} = \bar{U}_{S_3 \cup S_4} = \mathbf{0}$.
- $\|\mathbf{U}_\emptyset\|^2 = \|\mathbf{W}_{(\emptyset, \emptyset)}\|^2 = 1$.

Now we check that the balance condition (the last constraint in relaxation Ψ_1) is satisfied. We will prove that

$$\sum_v \bar{U}_{\{v\}} = (M+1)m\bar{U}_\emptyset,$$

Using Observation 2.10, we see that $\sum_{(C_i, \alpha) \in L_\Psi} \bar{U}_{\{(C_i, \alpha)\}} = \sum_{C_i} \bar{U}_\emptyset = m\bar{U}_\emptyset$. Similarly

$$\begin{aligned} \sum_{(x_j, \alpha, t) \in R_\Psi} \bar{U}_{\{(x_j, \alpha, t)\}} &= \sum_{j=1}^n \sum_{\alpha: \{x_j\} \rightarrow \{0,1\}} \sum_{t=1}^{\beta M} \bar{U}_{\{(x_j, \alpha, t)\}} = \beta M \cdot \sum_{j=1}^n \sum_{\alpha: \{x_j\} \rightarrow \{0,1\}} \bar{U}_{\{(x_j, \alpha)\}} \\ &= \beta M n \cdot \bar{U}_\emptyset = M m \bar{U}_\emptyset. \end{aligned}$$

Thus

$$\sum_{v \in V} \bar{U}_{\{v\}} = \sum_{v \in L_\Psi \cup R_\Psi \cup Z_r} \bar{U}_{\{v\}} = \sum_{(C_i, \alpha) \in L_\Psi} \bar{U}_{\{(C_i, \alpha)\}} + \sum_{(x_j, \alpha, t) \in R_\Psi} \bar{U}_{\{(x_j, \alpha, t)\}} = (M+1)m\bar{U}_\emptyset.$$

Now, we calculate the value of the solution

$$\begin{aligned} & \sum_{(u,v) \in \mathcal{E}_\Psi} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 \\ &= \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \sum_{z=1}^3 \left\| \bar{U}_{\{(C_i, \alpha)\}} - \bar{U}_{\{(x_{i_z}, \alpha|_{\{x_{i_z}\}}, 1)\}} \right\|^2 \\ & \quad + \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \sum_{v \in Z_r: ((C_i, \alpha), v) \in \mathcal{E}_\Psi} \|\bar{U}_{\{(C_i, \alpha)\}} - \bar{U}_{\{v\}}\|^2 \\ & \quad + \sum_{j=1}^n \sum_{\alpha: \{x_j\} \rightarrow \{0,1\}} \sum_{z_1, z_2 \in [M\beta]} \left\| \bar{U}_{\{(x_j, \alpha, z_1)\}} - \bar{U}_{\{(x_j, \alpha, z_2)\}} \right\|^2 + \sum_{v_1, v_2 \in Z_r} \|\bar{U}_{\{v_1\}} - \bar{U}_{\{v_2\}}\|^2 \\ &= \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \left(\sum_{z=1}^3 \left\| \bar{U}_{\{(C_i, \alpha)\}} - \bar{U}_{\{(x_{i_z}, \alpha|_{\{x_{i_z}\}}, 1)\}} \right\|^2 + 2 \|\bar{U}_{\{(C_i, \alpha)\}}\|^2 \right) \\ &= \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \left(\sum_{z=1}^3 \left\| \mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha)} - \mathbf{W}_{(\{x_{i_z}, \alpha|_{\{x_{i_z}\}}, 1)} \right\|^2 + 2 \left\| \mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha)} \right\|^2 \right) \\ &= \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \left(\sum_{z=1}^3 \left(\left\| \mathbf{W}_{(\{x_{i_z}, \alpha|_{\{x_{i_z}\}}, 1)} \right\|^2 - \left\| \mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha)} \right\|^2 \right) + 2 \left\| \mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha)} \right\|^2 \right) \\ &= \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \sum_{z=1}^3 \left\| \mathbf{W}_{(\{x_{i_z}, \alpha|_{\{x_{i_z}\}}, 1)} \right\|^2 - \sum_{i=1}^m \sum_{\alpha: \{x_{i_1}, x_{i_2}, x_{i_3}\} \rightarrow \{0,1\}, C_i(\alpha)=1} \left\| \mathbf{W}_{(\{x_{i_1}, x_{i_2}, x_{i_3}\}, \alpha)} \right\|^2 \\ &= \sum_{i=1}^m \sum_{z=1}^3 2 \left(\left\| \mathbf{W}_{\{x_{i_z}\}, \{x_{i_z} \rightarrow 0\}} \right\|^2 + \left\| \mathbf{W}_{\{x_{i_z}\}, \{x_{i_z} \rightarrow 1\}} \right\|^2 \right) - m \\ &= 6m - m = 5m. \end{aligned}$$

□

3.3 Soundness : bound for integral solutions

Let $\mathcal{L} = \{(x_j, \alpha) | \alpha : \{x_j\} \rightarrow \{0,1\}\}$ be the set of $2n$ literals. For each literal $(x_j, \alpha) \in \mathcal{L}$, let $\deg((x_j, \alpha))$ be the number of left vertices that connect to the literal's representative vertex $(x_j, \alpha, 1)$. For a set of literals $\mathcal{L}' \subseteq \mathcal{L}$, let $\deg(\mathcal{L}') = \sum_{(x_j, \alpha) \in \mathcal{L}'} \deg((x_j, \alpha))$. Also, given a subset $\mathcal{L}' \subseteq \mathcal{L}$, for left vertex (C_i, α) , say (C_i, α) is *contained* in \mathcal{L}' if all the three literals corresponding to the three neighbors of (C_i, α) in H_Ψ are contained in \mathcal{L}' , i.e. $\{(x_{i_1}, \alpha|_{x_{i_1}}), (x_{i_2}, \alpha|_{x_{i_2}}), (x_{i_3}, \alpha|_{x_{i_3}})\} \subseteq \mathcal{L}'$.

We first prove the following lemma regarding the structure of H_Ψ , defined by a random 3-XOR instance Ψ .

Lemma 3.3. *Over the choice of random 3-XOR instance Ψ , with probability $1 - o(1)$, the following statements hold.*

- For each $\mathcal{L}' \subseteq \mathcal{L}$, $|\mathcal{L}'| \geq n/3$, we have $\deg(\mathcal{L}') \geq 6m \cdot |\mathcal{L}'|/n(1 - 20/\sqrt{\beta})$.
- For each $\mathcal{L}' \subseteq \mathcal{L}$, $|\mathcal{L}'| \geq n/3$, the number of left vertices in L_Ψ contained in \mathcal{L}' is at most $m \cdot |\mathcal{L}'|^3/(2n^3) \cdot (1 + 100/\sqrt{\beta})$.

Proof. Fix a literal (x_j, α) , a random constraint C_i accesses x_j with probability $3/n$. Once C_i accesses x_j , there are 2 vertices out of the 4 left vertices corresponding to C_i adjacent to (x_j, α) . Therefore, in expectation, there are $6/n$ edges from the left vertices corresponding to C_i to (x_j, α) . By linearity of expectation, fix $\mathcal{L}' \subseteq \mathcal{L}$, there are $6|\mathcal{L}'|/n$ edges from the left vertices corresponding to a random constraint C_i to \mathcal{L}' in expectation.

By standard Chernoff bound, fix $\mathcal{L}' \subseteq \mathcal{L}$ such that $|\mathcal{L}'| \geq n/3$,

$$\begin{aligned} \Pr[\deg(\mathcal{L}') < 6m \cdot |\mathcal{L}'|/n(1 - 20/\sqrt{\beta})] &\leq \exp\left(-m \cdot |\mathcal{L}'|/(2n) \cdot (20/\sqrt{\beta})^2/2\right) \\ &\leq \exp(-\beta n/6 \cdot 200/\beta) \leq 2^{-4n}. \end{aligned}$$

Since there are at most 2^{2n} such \mathcal{L}' 's, by a union bound, with probability at least $1 - 2^{-2n}$, the first statement holds.

For the second statement, fix an $\mathcal{L}' \subseteq \mathcal{L}$, let a_0, a_1, a_2 be the number of variables that has 0, 1, 2 corresponding literals in \mathcal{L}' , respectively. Note that $a_0 + a_1 + a_2 = n$ and $a_1 + 2a_2 = |\mathcal{L}'|$. Now, for a random constraint C_i , we are interested in the expected number of the four corresponding left vertices (C_i, α) that are contained in \mathcal{L}' . Note that once C_i accesses a variable that corresponds to a_0 , none of the four corresponding left vertices are contained in \mathcal{L}' . Now assume that there are t of the 3 variables accessed by C_i have two literals in \mathcal{L}' and the other $(3 - t)$ variables have one literal in \mathcal{L}' . Observe that in expectation, there are 2^{t-1} left vertices corresponding to C_i contained in \mathcal{L}' .

In all, the expected number of the left vertices corresponding to C_i that are contained in \mathcal{L}' is

$$\begin{aligned} \sum_{t=0}^3 \frac{\binom{a_1}{3-t} \binom{a_2}{t}}{\binom{n}{3}} &< (1 + \frac{10}{n}) \sum_{t=0}^3 \binom{3}{t} (a_1/n)^{3-t} (a_2/n)^t \cdot 2^{t-1} \quad (\text{for } n > 3) \\ &= (1 + \frac{10}{n}) (a_1 + 2a_2)^3 / (2n^3) = (1 + \frac{10}{n}) \cdot |\mathcal{L}'|^3 / (2n^3). \end{aligned}$$

By standard Chernoff bound,

$$\begin{aligned} &\Pr[\text{\#left vertices contained in } \mathcal{L}' > m \cdot |\mathcal{L}'|^3 / (2n^3) \cdot (1 + 100/\sqrt{\beta})] \\ &\leq \exp\left(-m \cdot |\mathcal{L}'|^3 / (2n^3) / 4 \cdot ((100/\sqrt{\beta} - 10/n) \cdot n / (n + 10))^2 / 2\right) \\ &\leq \exp\left(-m \cdot |\mathcal{L}'|^3 / (2n^3) / 4 \cdot (80/\sqrt{\beta})^2 / 2\right) \quad (\text{for } n \gg \sqrt{\beta}) \\ &\leq \exp(-\beta n / 6^3 \cdot 3200/\beta) \leq 2^{-4n}. \end{aligned}$$

Since there are at most 2^{2n} such \mathcal{L}' 's, by a union bound, with probability at least $1 - 2^{-2n}$, the second statement holds. \square

Now, we are ready to prove the soundness lemma.

Lemma 3.4. *For $\tau > 1/3$, with probability $1 - o(1)$, the τ vs. $(1 - \tau)$ balanced separator has at least $4m(3\tau - \tau^3 - O(1/\sqrt{\beta}) - O(1/M))$ edges in the cut.*

Proof. We are going to prove that, once the two conditions in Lemma 3.3 hold, we have the desired upperbound for τ vs. $(1 - \tau)$ balanced separator. Let us assume that there is a balanced separator is (A', B') and $\text{edges}(A', B') \leq 4m(3\tau - \tau^3) \leq 12m$, we will show that $\text{edges}(A', B') \geq 4m(3\tau - \tau^3 - O(1/\sqrt{\beta}) - O(1/M))$.

Based on (A', B') we build another cut (A, B) such that $A \cap Z_r = A' \cap Z_r$ and $A \cap R_\Psi = A' \cap R_\Psi$. For each left vertex in L_Ψ , it has 5 edges going to Z_r and R_Ψ . We assign the vertex to A if it has less than 3 edges going to $B \cap (Z_r \cup R_\Psi)$, and assign it to B otherwise. Note that $\text{edges}(A, B) \leq \text{edges}(A', B')$, therefore we only need to show that $\text{edges}(A, B) \geq m(12\tau - \tau^3 - O(1/\sqrt{\beta}) - O(1/M))$. Since L_Ψ contains only $O(1/M)$ fraction of the total vertices, (A, B) is still $(\tau - O(1/M))$ vs. $(1 - \tau + O(1/M))$ balanced.

Since $\text{edges}(A, B) \leq 12m$, for large enough constant M , we have the following two statements.

- 1) One of $A \cap Z_r$ and $B \cap Z_r$ has at most $100/M \cdot |Z_r| = 100\sqrt{\beta n/M}$ vertices.
- 2) Let $\mathcal{C}_{\text{bad}} = \{(x_j, \alpha) : \text{the clique } C_{(x_j, \alpha)} \text{ is broken by } (A, B)\}$, then $|\mathcal{C}_{\text{bad}}| \leq 20n/M$.

If 1) does not hold, then we see there are at least $(100/M)(1 - 100/M) \cdot |Z_r|^2$ edges in Z_r cut by (A, B) , while for large enough M , this number is greater than $100/M \cdot 1/2 \cdot |Z_r|^2 = 50/M \cdot M\beta n = 50m$. If 2) does not hold, for each clique $C_{(x_j, \alpha)}$ that is broken by (A, B) , at least $(\beta M - 1)$ edges of the clique are on the cut. In all, there are at least $(\beta M - 1) \cdot 20n/M > 12\beta n = 12m$ edges in the cut.

Now, by 1), assume w.l.o.g. that $A \cap Z_r$ is the smaller side – having at most $100/M \cdot |Z_r|$ vertices, and let \mathcal{L}' be the set of literals (x_j, α) such that its representative vertex $(x_j, \alpha, 1)$ is in A .

To get a lower bound for $|\mathcal{L}'|$, note that

$$|A| \leq (|\mathcal{L}'| + |\mathcal{C}_{\text{bad}}|) \cdot M\beta + |Z_r| + |L_\Psi| = |\mathcal{L}'| \cdot M\beta + O(1)m. \quad (4)$$

Also, since (A, B) is a balanced separator, we have $|A| \geq (\tau - O(1/M)) \cdot 2Mm$. Hence, by (4), we have $|\mathcal{L}'| \geq (\tau - O(1/M)) \cdot 2n$.

Let $L_{\text{bad}} \subseteq L_\Psi$ be the set of left vertices that at least one of the two neighbors in Z_r falls into $A \cap Z_r$. By the regularity of graph where Z_r is set of vertices and L_Ψ is set of edges, we know that $|L_{\text{bad}}| \leq 8\sqrt{\beta n/M} \cdot 100/M \cdot |Z_r| \leq O(m/M)$.

Now let us get a lower bound on $\text{edges}(A, B)$. First, we have $\text{edges}(A, B) \geq \text{edges}(A \setminus L_{\text{bad}}, B \setminus L_{\text{bad}})$. Let $L'_\Psi = L_\Psi \setminus L_{\text{bad}}$, we have

$$\begin{aligned} & \text{edges}(A \setminus L_{\text{bad}}, B \setminus L_{\text{bad}}) \\ &= \text{edges}(A \cap (L'_\Psi \cup R_\Psi \cup Z_r), B \cap (L'_\Psi \cup R_\Psi \cup Z_r)) \\ &\geq \text{edges}(A \cap L'_\Psi, B \cap Z_r) + \text{edges}(A \cap R_\Psi, B \cap L'_\Psi) \\ &= \text{edges}(A \cap L'_\Psi, B \cap Z_r) + \text{edges}(A \cap R_\Psi, L'_\Psi) - \text{edges}(A \cap R_\Psi, A \cap L'_\Psi) \\ &\geq \text{edges}(A \cap L'_\Psi, B \cap Z_r) + \text{edges}(A \cap R_\Psi, L_\Psi) - |L_{\text{bad}}| \cdot 3 - \text{edges}(A \cap R_\Psi, A \cap L'_\Psi). \end{aligned}$$

For left vertices $(C_i, \alpha) \in L'_\Psi$, we claim that it is contained in \mathcal{L}' if and only if $(C_i, \alpha) \in A$. This is because if it is contained in \mathcal{L}' , then we have $(C_i, \alpha) \in A$ because 3 out of 5 edges incident to

(C_i, α) go to A side (the three variable representative vertices). If (C_i, α) is not contained in \mathcal{L}' , we have at least 3 out of the 5 edges going to B side (the two edges to $B \cap Z_r$ and at least one of the variable representative vertices), and therefore we have $(C_i, \alpha) \in B$. By this claim, we know the following two facts.

- $|A \cap L'_\Psi|$ is small. Since $\tau > 1/3$, we have $|\mathcal{L}'| \geq (2/3 - O(1/M))n > n/3$, by the second property of [Lemma 3.3](#), we have $|A \cap L'_\Psi| \leq m \cdot |\mathcal{L}'|^3/(2n^3) \cdot (1 + 80/\sqrt{\beta})$.
- We have $\text{edges}(A \cap L'_\Psi, B \cap Z_r) = 2|A \cap L'_\Psi|$ and $\text{edges}(A \cap L'_\Psi, A \cap R_\Psi) = 3|A \cap L'_\Psi|$.

For $\text{edges}(A \cap R_\Psi, L_\Psi)$, we know that this is exactly $\deg(\mathcal{L}')$. Again, since $\tau > 1/3$, by the first property of [Lemma 3.3](#), we know this value is lower-bounded by $6m \cdot |\mathcal{L}'|/n(1 - 20/\sqrt{\beta})$.

In all, we have

$$\begin{aligned}
\text{edges}(A, B) &\geq \text{edges}(A \cap L'_\Psi, B \cap Z_r) + \text{edges}(A \cap R_\Psi, L_\Psi) - |L_{\text{bad}}| \cdot 3 - \text{edges}(A \cap R_\Psi, A \cap L'_\Psi) \\
&= 2|A \cap L'_\Psi| + \deg(\mathcal{L}') - |L_{\text{bad}}| \cdot 3 - 3|A \cap L'_\Psi| \\
&\geq \deg(\mathcal{L}') - |A \cap L'_\Psi| - O(m/M) \\
&\geq 6m \cdot |\mathcal{L}'|/n(1 - 20/\sqrt{\beta}) - m \cdot |\mathcal{L}'|^3/(2n^3) \cdot (1 + 80/\sqrt{\beta}) - O(m/M) \\
&= m \left(12\gamma - 4\gamma^3 - 20(12\gamma - 4\gamma^3)/\sqrt{\beta} - O(1/M) \right) \quad (\text{let } \gamma = |\mathcal{L}'|/(2n)) \\
&\geq 4m \left(3\tau - \tau^3 - O(1/\sqrt{\beta}) - O(1/M) \right).
\end{aligned}$$

The last step is because i) $3\gamma - \gamma^3$ monotonically increases when $\gamma \in [0, 1]$, and ii) $\gamma \geq (\tau - O(1/M))$. \square

4 Gaps for Uniform Sparsest Cut

In this section, we modify the gap instance we got for [Balanced Separator](#) to get an integrality gap for the linear round Lasserre relaxation of [Uniform Sparsest Cut](#).

The reduction converts the gap instance for [Balanced Separator](#) to the gap instance for [Uniform Sparsest Cut](#) in an almost black box style. In the [Balanced Separator](#) problem, we have the hard constraint that the cut is τ -balanced. In the reduction from [Balanced Separator](#) to [Uniform Sparsest Cut](#), we need to use the sparsity objective to enforce this constraint. We do it as follows. Recall that given a 3-XOR instance Ψ , the corresponding gap instance for [Balanced Separator](#) consists of vertex set $L_\Psi \cup R_\Psi \cup Z_r$ and edge set \mathcal{E}_Ψ . To get a gap instance for [Uniform Sparsest Cut](#), we add two more cliques D_l and D_r of size $1000Mm$ (where M is the same parameter defined in the previous sections). Now, let the edge set \mathcal{E}'_Ψ contain the edges in \mathcal{E}_Ψ , in the cliques D_l and D_r , and the following edges : for each vertex $v \in L_\Psi \cup R_\Psi \cup Z_r$, introduce 2 new edges incident to it, one to an arbitrary vertex in D_l (say, v_l) and the other one to an arbitrary vertex in D_r (say, v_r).

Using the instance described above, we will prove [Theorem 1.3](#), which is stated in detail as follows.

Theorem 4.1. *For large enough constant β, M (where β is the same parameter as in previous sections), there is an instance for [Uniform Sparsest Cut](#) problem, such that the optimal solution is at least $(1 + 1/(100M))$ times worse the optimal solution of the $\Omega(n)$ -round Lasserre SDP.*

Proof of Completeness. We show that the value of relaxation Ψ_2 (in Definition 2.4) is at most $(2M + 9 + \sqrt{M/m})m/((1001M + 1)m)^2$ for $\tau = (1001M + 1)/(2002M + 4 + \sqrt{M/m})$.

Given the SDP solution $\{\bar{U}_{S'}\}_{S' \subseteq L_\Psi \cup R_\Psi \cup Z_r, |S'| \leq r/3}$ in the completeness case of Balanced Separator, we extend it to the SDP solution $\{\bar{U}_S\}_{S \subseteq L_\Psi \cup R_\Psi \cup Z_r \cup D_l \cup D_r, |S| \leq r/3}$ for Uniform Sparsest Cut by “putting D_l and D_r one per side”. That is, for each $S \subseteq L_\Psi \cup R_\Psi \cup Z_r \cup D_l \cup D_r$ with $|S| \leq r/3$, let $S' = S \cap (L_\Psi \cup R_\Psi \cup Z_r)$. Now we let $\bar{U}_S = \mathbf{0}$ if $S \cap D_r \neq \emptyset$, and let $\bar{U}_S = \bar{U}_{S'}$ otherwise.

We first check that $\{\bar{U}_S\}_{S \subseteq L_\Psi \cup R_\Psi \cup Z_r \cup D_l \cup D_r, |S| \leq r/3}$ is a feasible SDP solution. We only check that the balance constraint (the last constraint in relaxation Ψ_2) is met.

We are going to prove that

$$\sum_{u \in L_\Psi \cup R_\Psi \cup Z_r \cup D_l \cup D_r} \bar{U}_{\{u\}} = (1001M + 1)m\bar{U}_\emptyset.$$

From the proof of Lemma 3.2, we know that

$$\sum_{u \in L_\Psi \cup R_\Psi \cup Z_r} \bar{U}_{\{u\}} = (M + 1)m\bar{U}_\emptyset,$$

together with the fact that

$$\forall u \in D_l, \bar{U}_{\{u\}} = \bar{U}_\emptyset, \quad \forall u \in D_r, \bar{U}_{\{u\}} = \mathbf{0},$$

we get the desired equality.

Now we calculate the value of the solution. First, we calculate the following value.

$$\begin{aligned} \sum_{(u,v) \in \mathcal{E}'_\Psi} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 &= \sum_{(u,v) \in \mathcal{E}_\Psi} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 + \sum_{(u,v) \in \mathcal{E}'_\Psi \setminus \mathcal{E}_\Psi} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 \\ &= 5m + \sum_{u,v \in D_l} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 + \sum_{u,v \in D_r} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 \\ &\quad + \sum_{u \in L_\Psi \cup R_\Psi \cup Z_r} \left(\|\bar{U}_{\{u\}} - \bar{U}_{\{v_l\}}\|^2 + \|\bar{U}_{\{u\}} - \bar{U}_{\{v_r\}}\|^2 \right), \end{aligned}$$

Note that $\sum_{u,v \in D_l} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 + \sum_{u,v \in D_r} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 = 0$, and

$$\begin{aligned} &\sum_{u \in L_\Psi \cup R_\Psi \cup Z_r} \left(\|\bar{U}_{\{u\}} - \bar{U}_{\{v_l\}}\|^2 + \|\bar{U}_{\{u\}} - \bar{U}_{\{v_r\}}\|^2 \right) \\ &= \sum_{u \in L_\Psi \cup R_\Psi \cup Z_r} \left(2\|\bar{U}_{\{u\}}\|^2 + \|\bar{U}_{\{v_l\}}\|^2 + \|\bar{U}_{\{v_r\}}\|^2 - 2\langle \bar{U}_{\{u\}}, \bar{U}_{\{v_l\}} \rangle - 2\langle \bar{U}_{\{u\}}, \bar{U}_{\{v_r\}} \rangle \right) \\ &= \sum_{u \in L_\Psi \cup R_\Psi \cup Z_r} \left(2\|\bar{U}_{\{u\}}\|^2 + 1 + 0 - 2\|\bar{U}_{\{u, v_l\}}\|^2 - 2\|\bar{U}_{\{u, v_r\}}\|^2 \right) \quad (\text{by property of Lasserre vectors}) \\ &= \sum_{u \in L_\Psi \cup R_\Psi \cup Z_r} 1 = |L_\Psi| + |R_\Psi| + |Z_r| = (2M + 4 + \sqrt{M/m})m. \end{aligned}$$

Thus, we have

$$\sum_{(u,v) \in \mathcal{E}'_\Psi} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 = (2M + 9 + \sqrt{M/m})m.$$

Since $\tau < 1/2$, the value of the solution is at most

$$\frac{1}{|L_\Psi \cup R_\Psi \cup Z_r \cup D_l \cup D_r|^{2\tau^2}} \sum_{(u,v) \in \mathcal{E}'_{\Psi}} \|\bar{U}_{\{u\}} - \bar{U}_{\{v\}}\|^2 = (2M + 9 + \sqrt{M/m})m / ((1001M + 1)m)^2.$$

□

Proof of Soundness. We prove that for large enough M , the sparsity of the sparsest cut is at least $\gamma = (1 + 1/(100M)) \cdot (2M + 9 + \sqrt{M/m})m / (1001Mm)^2$.

First, we show that to get a sparse cut of sparsity better than γ , the clique D_l cannot be separated, and the same is true for D_r (by the same argument). This is because if D_l is separated, there are at least $(1000Mm - 1)$ edges on the cut. Since the graph has $|L_\Psi| + |R_\Psi| + |Z_r| + |D_l| + |D_r| = (2002M + 4 + \sqrt{M/m})m$ vertices, therefore the sparsity of the cut is at least

$$\frac{1001Mm - 1}{\frac{1}{4} \cdot ((2002M + 4 + \sqrt{M/m})m)^2} > \frac{500Mm}{(1001Mm)^2} > \gamma,$$

for $M > 10$.

Second, we show that D_l and D_r should be on opposite sides of any cut of sparsity better than γ . Suppose not, let S be the side of the cut which D_l and D_r are not on. We know that $S \subseteq L_\Psi \cup R_\Psi \cup Z_r$ and therefore every vertex in S has 2 edges connected to D_l and D_r . Therefore $\text{edges}(S, \bar{S}) \geq 2|S|$. We upperbound $|\bar{S}|$ by the total number of vertices, which is $(2002M + 4 + \sqrt{M/m})m \leq (2002M + 5)m$ (for $m > M$). Therefore, for large enough M , we have

$$\frac{\text{edges}(S, \bar{S})}{|S||\bar{S}|} \geq \frac{2|S|}{|S| \cdot ((2002M + 5)m)} = \frac{1}{(1001M + 2.5)m} > \gamma.$$

Now, we show that if the cut (S, \bar{S}) has sparsity better than γ , we let (T, \bar{T}) be the cut restricted to $L_\Psi \cup R_\Psi \cup Z_r$ (the **Balanced Separator** instance), it is a 0.49 vs 0.51 balanced cut, i.e. $|T|/(|L_\Psi| + |R_\Psi| + |Z_r|) \in [0.49, 0.51]$. Suppose (T, \bar{T}) is not 0.49 vs 0.51 balanced, we know that (S, \bar{S}) is not $0.5 - 10^{-5}$ vs $0.5 + 10^{-5}$ balanced, and therefore

$$|S||\bar{S}| < ((2002M + 4 + \sqrt{M/m})m)^2 \cdot (0.5 - 10^{-5})(0.5 + 10^{-5}) < (1001Mm)^2 \cdot (1 - 10^{-10}).$$

Since D_l and D_r are on opposite sides of (S, \bar{S}) , we know that $\text{edges}(S, \bar{S}) \geq (2M + 4 + \sqrt{M/m})m$, and therefore the sparsity of the cut

$$\frac{\text{edges}(S, \bar{S})}{|S||\bar{S}|} > \frac{(2M + 4 + \sqrt{M/m})m}{(1001Mm)^2} \cdot (1 + 10^{-10}).$$

This value is greater than γ when $M > 10^{20}$.

Finally, since (T, \bar{T}) is a 0.49 vs 0.51 balanced cut, by [Lemma 3.4](#), we know that with probability

$1 - o(1)$, $\text{edges}(T, \bar{T}) > (5.4 - O(1/\sqrt{\beta}) - O(1/M))m$. Therefore

$$\begin{aligned}
\frac{\text{edges}(S, \bar{S})}{|S||\bar{S}|} &\geq \frac{\text{edges}(T, \bar{T}) + (2M + 4 + \sqrt{M/m})m}{\frac{1}{4} \cdot ((2002M + 4 + \sqrt{M/m})m)^2} \\
&\geq \frac{(2M + 9.4 + \sqrt{M/m} - O(1/\sqrt{\beta}) - O(1/M))m}{\frac{1}{4} \cdot ((2002M + 4 + \sqrt{M/m})m)^2} \\
&\geq \frac{(2M + 9.4 + \sqrt{M/m} - O(1/\sqrt{\beta}) - O(1/M))m}{\frac{1}{4} \cdot ((2002M + 5)m)^2} \quad (\text{for } m > M) \\
&\geq \frac{(2M + 9.4 + \sqrt{M/m} - O(1/\sqrt{\beta}) - O(1/M))m}{(1001Mm)^2} \cdot (1 - 1/(200M)) \\
&\geq \frac{(2M + 9.1 + \sqrt{M/m})m}{(1001Mm)^2} \cdot (1 - 1/(200M)) \quad (\text{for large enough } \beta \text{ and } M) \\
&\geq \frac{(2M + 9 + \sqrt{M/m})m}{(1001Mm)^2} \cdot (1 + 1/(30M))(1 - 1/(200M)) \quad (\text{for large enough } M) \\
&\geq \frac{(2M + 9 + \sqrt{M/m})m}{(1001Mm)^2} \cdot (1 + 1/(100M)) = \gamma.
\end{aligned}$$

□

5 Gaps for Maximum Cut

In this section, we prove [Theorem 1.4](#), which is restated formally as follows.

Theorem 5.1. *There exists an absolute constant $\eta > 0$ such that there is an instance for Maximum Cut problem such that the value of the optimal cut is at most $\frac{17}{18} + o(1)$ times the optimal value of the ηn -round Lasserre SDP relaxation (where n is the number of vertices in the graph).*

Proof. For this case, we will instead reduce from a random Monotone 4-XOR instance Ψ with $m = \beta n$ constraints and n variables and use the construction provided by [Theorem 2.9](#). As a reminder, our constraints are of the form:

$$x_{i_1} \oplus x_{i_2} \oplus x_{i_3} \oplus x_{i_4} = 1. \quad (5)$$

A simple probabilistic calculation shows that for large enough β , no Boolean assignment will satisfy more than $\frac{m}{2}(1 + o(1))$ of the constraints.

We now describe the reduction to Maximum Cut. Again we build an almost bipartite graph $\mathcal{H}_\Psi = (L_\Psi, R_\Psi, \mathcal{E}_\Psi)$ as follows. Left side L_Ψ of \mathcal{H}_Ψ contains a vertex for each variable and right side R_Ψ contains a vertex for each constraint, so that $|L_\Psi| = n, |R_\Psi| = m$.

For every constraint $C_i = \{i_1, i_2, i_3, i_4\} \in \Psi$ of the form (5), we add the following set of edges, E_{C_i} with corresponding weights $w_{C_i} : E_{C_i} \rightarrow \mathbb{R}_+$ to the graph.

- There is a clique between nodes $\{i_1, i_2, i_3, i_4\} \subset L_\Psi$ whose all edges have unit weight.
- There is an edge from node $i \in R_\Psi$ corresponding to constraint C_i , to all nodes $\{i_1, i_2, i_3, i_4\}$ with weight 2.

After we are done, we will remove parallel edges by summing up their weights so that our final instance will be a weighted simple graph with edge $e = \{u, v\} \in \mathcal{E}_\Psi$ having weight w_e . First we prove the following properties of this gadget:

Claim 5.2 (Gadget Completeness). *For given $C_i = \{i_1, \dots, i_4\}$, let $\alpha : \{i_1, \dots, i_4, C_i\} \rightarrow \{0, 1\}$ be an assignment such that*

$$\alpha(i_1) \oplus \dots \oplus \alpha(i_4) = 1,$$

and $\alpha(C_i)$ is equal to the minority of $\alpha(i_1), \dots, \alpha(i_4)$. Then the cut produced by α has weight at least 9.

Proof. By case analysis on $\sum_j \alpha(i_j) \in \{1, 3\}$. If $\sum_j \alpha(i_j) = 1$, then $\alpha(C_i) = 1$. Hence weight of cut is $3 \cdot 2 + 3 = 9$. If $\sum_j \alpha(i_j) = 3$, then $\alpha(C_i) = 0$ whose cut value is 9. \square

Claim 5.3 (Gadget Soundness). *For given $C_i = \{i_1, \dots, i_4\}$, let $\alpha : \{i_1, \dots, i_4, C_i\} \rightarrow \{0, 1\}$ be an assignment with*

$$\alpha(i_1) \oplus \dots \oplus \alpha(i_4) = 0.$$

Then its cut weight is at most 8.

Proof. $\alpha(i_1) \oplus \dots \oplus \alpha(i_4) = 0$ implies that $\sum_j \alpha(i_j) \in \{0, 2\}$. If $\sum_j \alpha(i_j) = 0$, then cut value is maximized only if $\alpha(C_i) = 1$, in which case it is equal to 8. If $\sum_j \alpha(i_j) = 2$, then regardless of the value of $\alpha(C_i)$, the cut value is always 8. \square

Completeness for Integrality Gap. Finally we only need to exhibit vectors representing the *minority* assignment to nodes for each constraint. For each set $S \subseteq L_\Psi \cup R_\Psi$ with $|S| \leq r/4$ and $S \cap R_\Psi = \{C_1, C_2, \dots, C_{r_1}\}$, we define the vector \overline{U}_S as follows.

Define $T(S)$ to be the set of all variables appearing in constraints $\{C_1, \dots, C_{r_1}\} \subseteq S$ and in $S \cap L_\Psi$. Let $\mathcal{A}(S)$ be the following set of assignments to all variables in set $T(S)$:

$$\mathcal{A}(S) \triangleq \left\{ \alpha : T(S) \rightarrow \{0, 1\} \mid \text{for all } C_i \text{ in } S, \text{ minority of } (\alpha(j))_{j \in C_i} \text{ is } 1. \right\}$$

Observe that each node $C_i \in R_\Psi$ depends locally on its variables. Therefore for any two set S_1 and S_2 , we can verify the following:

$$\mathcal{A}(S_1 \cup S_2) = \left\{ \alpha : T(S_1 \cup S_2) \rightarrow \{0, 1\} \mid \alpha(T(S_1)) \in \mathcal{A}(S_1) \wedge \alpha(T(S_2)) \in \mathcal{A}(S_2) \right\}.$$

We set $\overline{U}_S = \sum_{\alpha \in \mathcal{A}(S)} \mathbf{W}_{(T(S), \alpha)}$. Note that for each constraint, $|C_i| \leq 4$, hence $|T| \leq 4|S| \leq r$. We only need to verify the consistency constraint:

$$\begin{aligned} \langle \overline{U}_{S_1}, \overline{U}_{S_2} \rangle &= \sum_{\alpha_1 \in \mathcal{A}(S_1), \alpha_2 \in \mathcal{A}(S_2)} \langle \mathbf{W}_{(T(S_1), \alpha_1)}, \mathbf{W}_{(T(S_2), \alpha_2)} \rangle \\ &= \sum_{\substack{\alpha_1 \in \mathcal{A}(S_1), \alpha_2 \in \mathcal{A}(S_2) \\ \alpha_1(T(S_1) \cap T(S_2)) = \alpha_2(T(S_1) \cap T(S_2))}} \langle \mathbf{W}_{(T(S_1), \alpha_1)}, \mathbf{W}_{(T(S_2), \alpha_2)} \rangle \\ &= \sum_{\alpha \in \mathcal{A}(S_1 \cup S_2)} \|\mathbf{W}_{(T(S_1 \cup S_2), \alpha)}\|^2 = \|\overline{U}_{S_1 \cup S_2}\|^2. \end{aligned}$$

The objective value for this construction immediately follows from gadget completeness. For each constraint C_i , consider the subset $S = \{i_1, i_2, i_3, i_4, C_i\}$ which includes all nodes corresponding variables from L_Ψ and the node from R_Ψ corresponding to this constraint, so that $T(S) = S$. Since vectors \bar{U} satisfy Lasserre constraints, we know that for any j , $\bar{U}_{i_j} = \sum_{\alpha \in \mathcal{A}(S), \alpha(i_j)=1} \mathbf{W}_{(S,\alpha)}$ and $\bar{U}_{C_i} = \sum_{\alpha \in \mathcal{A}(S)} \mathbf{W}_{(S,\alpha)}$. Consequently, the objective function restricted to edges of E_{C_i} is:

$$\begin{aligned} \sum_{(p,q) \in E_{C_i}} w_{p,q} \|\bar{U}_p - \bar{U}_q\|^2 &= \sum_{\alpha: \{i_1, \dots, i_4\} \rightarrow \{0,1\}, \beta = \text{minority of } (\alpha)} \|\mathbf{W}_{(S,\alpha)}\|^2 \text{weight of edges cut by } (\alpha, \beta) \\ &= \sum_{\alpha(i_1) \oplus \dots \oplus \alpha(i_4) = 1, \beta = \text{minority of } (\alpha)} \|\mathbf{W}_{(S,\alpha)}\|^2 \text{weight of edges cut by } (\alpha, \beta) \\ &\geq 9 \sum_{\alpha(i_1) \oplus \dots \oplus \alpha(i_4) = 1} \|\mathbf{W}_{(S,\alpha)}\|^2 = 9. \end{aligned}$$

Hence the objective value is at least $9m$.

Since Ψ has no assignment satisfying more than $\frac{m}{2}(1 + o(1))$ many constraints, any cut in the graph \mathcal{H}_Ψ will cut at most

$$(1 + o(1))\left(\frac{m}{2}9 + \frac{m}{2}8\right) = \frac{17}{2}m(1 + o(1))$$

many edges as shown in the gadget soundness analysis. \square

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